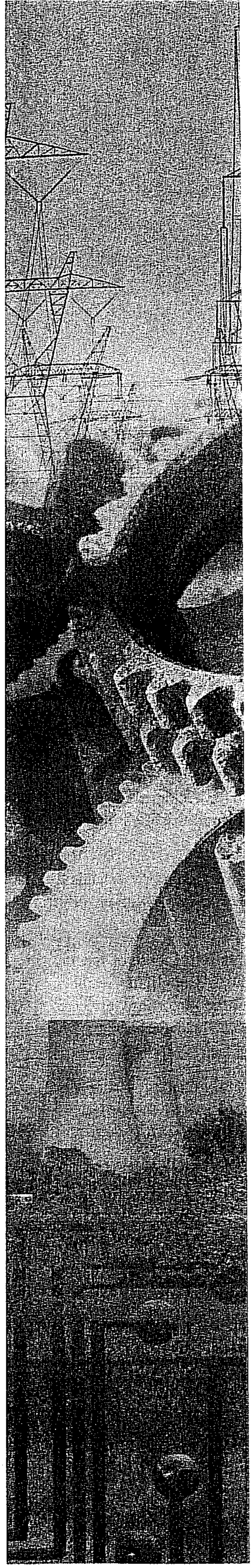

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Compressible Flow

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36.1 Introduction

Compressible flow is defined as *variable density flow*; this is in contrast to incompressible flow, where the density is assumed to be constant throughout. The variation in density is mainly caused by variations in pressure and temperature. We sometimes call the study of such fluids in motion **gas dynamics.** Fluid compressibility is a very important consideration in modern engineering applications. Knowledge of compressible fluid flow theory is required in the design and operation of many devices commonly encountered in engineering practice. A few important examples are the external flow over modern high-speed aircrafts; internal flows through rocket, gas turbine, and reciprocating engines; flow through natural gas transmission pipelines; and flow in high-speed wind tunnels.

The variation of fluid density and other fluid properties for compressible flow gives rise to the occurrence of strange phenomena in compressible flow not found in incompressible flow. For example, with compressible flows we can have fluid deceleration in a convergent duct, fluid temperature decrease with heating, fluid acceleration due to friction, and discontinuous property changes in the flow.

There are many useful compressible flow references that the reader can consult, such as Anderson [2003], Oosthuizen and Carscallen [1997], Hodge and Koenig [1995], John [1984], and Zucrow and Hoffman [1976]. The objectives of this chapter are to primarily study compressibility effects by considering the steady, one-dimensional flow of an ideal gas. Although many real flows of engineering interest are more complex, these restrictions will allow us to concentrate on the effects of basic flow processes. Another aspect of this chapter is the consistent formulation of the equations in a form suitable for computer solution. The author and his associates have developed interactive software for the calculation of the properties of various compressible flows. The first version of the software called ***COMPROP*** was developed to accompany the textbook by Oosthuizen and Carscallen [1997]. A more recent version of the software called ***COMPROP2*** was developed to accompany the recent textbook by Anderson [2003]. For more detail about the development of ***COMPROP2*** and its capabilities see Tam et al. [2001].

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36.2 The Mach Number and Flow Regimes

The single most important parameter in the analysis of the compressible fluids is the **Mach Number** (M), named after the nineteenth century Austrian physicist Ernst Mach. The Mach number (a dimensionless measure of compressibility) is defined as:

$$M = \frac{V}{a} \tag{36.1}$$

where V is the local flow velocity and a is the local **speed of sound** (the other common symbol used for the local speed of sound is "c"). For an ideal gas the speed of sound is given by [Anderson, 2003]:

$$a = \sqrt{\gamma R T} \tag{36.2}$$

where γ is the specific heat ratio (= 1.4 for air), R is the gas constant (= 287 J/kg·K for air), and T is the absolute fluid temperature. The speed of sound in a gas depends, therefore, only on the absolute temperature of the gas. For air at standard sea level conditions the speed sound is about 341 m/sec.

The Mach number can be used to characterize flow regimes as follows (the numerical values listed are only rough guides):

Incompressible flow — The Mach number is very small compared to unity ($M < 0.3$). For practical purposes the flow is treated as incompressible. For air at standard sea level conditions this assumption is good for local flow velocities of about 100 m/sec or less.

Subsonic flow — The Mach number is less than unity but large enough so that compressible flow effects are present ($0.3 < M < 1$).

Sonic flow — The Mach number is unity ($M = 1$). The significance of the point at which Mach number is equal to 1 will be demonstrated in upcoming sections.

Transonic flow — The Mach number is very close to unity ($0.8 < M < 1.2$). Modern aircrafts are mainly powered by gas turbine engines that involve transonic flows.

Supersonic flow — The Mach number is larger than unity ($M > 1$). For this flow, a shock wave is encountered. There are dramatic differences (physical and mathematical) between subsonic and supersonic flows, as will be discussed in the future sections.

Hypersonic flow — The Mach number is larger than five ($M > 5$). When a space shuttle reenters the earth's atmosphere, the flow is hypersonic. At very high Mach numbers the flowfield becomes very hot and dissociation and ionization of gases take place. In these cases the assumption of an ideal gas is no longer valid, and the flow must be analyzed by the use of kinetic theory of gases rather than continuum mechanics.

In the development of the equations of the motion of a compressible fluid, much of the analysis will appear in terms of the Mach number.

36.3 Ideal Gas Relations

Before we can proceed with the development of the equations of the motion of a compressible flow, we need to become familiar with the ideal gas fluid we will be working with. The ideal gas property changes can be evaluated from the following **equation of state** for an ideal gas:

$$p = \rho R T \tag{36.3}$$

where p is the fluid absolute pressure, ρ is the fluid density, T is the fluid absolute temperature, and R is the gas constant (= 287 J/kg·K for air). The **gas constant**, R, represents a constant for each distinct ideal gas, where

$$R = \frac{\overline{R}}{M} \tag{36.4}$$

with this notation, $\overline{R}$ is the universal gas constant (= 8314 J/kg·mol·K) and M is the molecular weight of the ideal gas (= 28.97 for air).

For an ideal gas, **internal energy**, u, and **enthalpy**, h, are considered to be functions of temperature only, and where the **specific heats at constant volume and pressure**, c_v and c_p, are also functions of temperature only. The changes in the internal energy and enthalpy of an ideal gas are computed for constant specific heats as:

$$u_2 - u_1 = c_v(T_2 - T_1) \tag{36.5}$$

$$h_2 - h_1 = c_p(T_2 - T_1) \tag{36.6}$$

For variable specific heats one must integrate du = $\int c_v dT$ and $dh = \int c_p dT$ or use the gas tables [Moran and Shapiro, 2000]. Most modern thermodynamics texts now contain software for evaluating properties of nonideal gases [Çengel and Boles, 2002].

From Equation (36.5) and Equation (36.6), we see that changes in internal energy and enthalpy are related to the changes in temperature by the values of c_v and c_p. We will now develop useful relations for determining c_v and c_p. From Equation (36.5), Equation (36.6), and the definition of enthalpy (h = u + pv = u + RT) it can be shown that [Moran and Shapiro, 2000]:

$$c_p - c_v = R \tag{36.7}$$

Equation (36.7) indicates that the difference between c_v and c_p is constant for each ideal gas regardless of temperature. Also $c_p > c_v$. If the **specific heat ratio**, γ, is defined as (the other common symbol used for specific heat ratio is "k")

$$\gamma = \frac{c_p}{c_v} \tag{36.8}$$

then combining Equation (36.7) and Equation (36.8) leads to

$$c_p = \frac{\gamma R}{\gamma - 1} \tag{36.9}$$

and

$$c_v = \frac{R}{\gamma - 1} \tag{36.10}$$

For air at standard conditions, c_p = 1005 J/kg·K and c_v = 718 J/kg·K. Equation (36.9) and Equation (36.10) will be useful in our subsequent treatment of compressible flow.

For compressible flows, changes in the thermodynamic property **entropy**, s, are also important. From the first and the second laws of thermodynamics, it can be shown that the change in entropy of an ideal gas with constant specific heat values (c_v and c_p) can be obtained from [Anderson, 2003]:

$$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1} \tag{36.11}$$

and

$$s_2 - s_1 = c_v \ln\frac{T_2}{T_1} + R\ln\frac{v_2}{v_1} \tag{36.12}$$

For variable specific heats one must integrate $\int c_p dT$ and $\int c_v dT$ or use the gas tables [Moran and Shapiro, 2000]. Equation (36.11) and Equation (36.12) allow the calculation of the change in entropy of an ideal gas between two states with constant specific heat values in terms of either the temperature and pressure, or the temperature and specific volume. Note that entropy is a function of both T and p, or T and v but not temperature alone (unlike internal energy and enthalpy).

36.4 Isentropic Flow Relations

An adiabatic flow (no heat transfer) which is frictionless (ideal or reversible) is referred to as **isentropic** (constant entropy) flow. Such flow does not occur in nature. However, the actual changes experienced by the large regions of the compressible flow field are often well approximated by this process. This is the case in internal flows such as for nozzles and external flows such as around an airfoil. In the regions adjacent to the nozzle walls or the airfoil surface, a thin boundary layer is formed and isentropic flow approximation fails. In this region flow is not adiabatic and reversible which causes the entropy to increase in the boundary layer.

Important relations for an isentropic flow of an ideal gas with constant c_v and c_p can be obtained directly from Equation (36.11) and Equation (36.12) by setting the left-hand side of these equations to zero ($s_2 = s_1$)

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)} \tag{36.13}$$

Equation (36.13) relates absolute pressure, density, and absolute temperature for an isentropic process, and is very frequently used in the analysis of compressible flows.

36.5 Stagnation State and Properties

Stagnation state is defined as a state that would be reached by a fluid if it were brought to rest isentropically (reversibly and adaibatically) and without work. Figure 36.1 shows a stagnation point in compressible flow. The properties at the stagnation state are refereed to as **stagnation properties** (or total properties). The stagnation state and the stagnation properties are designated by the subscript 0 (or t). Stagnation properties are very useful and are used as a reference state for compressible flows.

Consider the steady flow of a fluid through a duct such as a nozzle, diffuser, or some other flow passage where the flow takes place adiabatically and with no shaft or electrical work. Assuming the fluid experi-

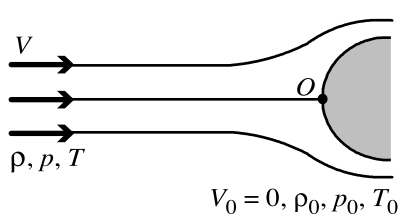

FIGURE 36.1 Stagnation point.

ences little or no change in its elevation and its potential energy, the energy equation between any two points in the flow for this single-stream steady-flow system reduces to

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} \tag{36.14}$$

In Equation (36.14), if we let one of the points to be stagnation point (V = 0), then

$$h_0 = h + \frac{V^2}{2} \tag{36.15}$$

where h_0 is the **stagnation enthalpy** and h is the **static enthalpy** of the fluid. Combining Equation (36.14) and Equation (36.15)

$$h_{01} = h_{02} \tag{36.16}$$

That is, in the absence of any heat and work interactions and any changes in potential energy, the stagnation enthalpy of a fluid remains constant during a steady-flow process. Flows through nozzles and diffusers usually satisfy these conditions, and any increase (or decrease) in fluid velocity in these devices will create an equivalent decrease (or increase) in the static enthalpy of the fluid.

Frequently, there is difficulty in understanding the difference between stagnation and static properties. Stagnation properties are those properties experienced by a fixed observer, the fluid being brought to rest at the observer (V = 0). **Static properties** are those properties experienced by an observer moving with the same velocity as the stream. The difference between the static and stagnation properties is due to the velocity (or kinetic energy) of the flow, see Equation (36.15). We may regard the stagnation conditions as local fluid properties. Aside from analytical convenience, the definition of the stagnation state is useful experimentally, since stagnation temperature, T_0, and stagnation pressure, p_0, are relatively easily measured. It is usually much more convenient to measure stagnation temperature T_0 than the static temperature T.

36.6 Stagnation Property Relations

Recall the definition of stagnation enthalpy given by Equation (36.15), for an ideal gas with constant specific heats, its static and stagnation enthalpies can be replaced by c_pT or c_pT_0, respectively,

$$c_p T_0 = c_p T + \frac{V^2}{2}$$

or

$$T_0 = T + \frac{V^2}{2c_p} \tag{36.17}$$

In Equation (36.17), the **stagnation temperature**, T_0, represents the temperature an ideal gas will attain when it is brought to rest adiabatically. The term $V^2/2c_p$ corresponds to the temperature rise during such a process and is called the **dynamic temperature** (or impact temperature rise). Note that for low-speed flows, the stagnation and static temperatures are typically the same. But for high-speed flows, the stagnation temperature (measured by a stationary probe, for example) may be significantly higher than the static temperature of the fluid.

Introducing Equation (36.1) for M and Equation (36.9) for c_p into Equation (36.17) we obtain

$$\frac{T_0}{T} = \left(1 + \frac{\gamma - 1}{2} M^2\right) \tag{36.18}$$

Equation (36.18) gives the ratio of the stagnation to static temperature at a point in a flow as a function of the Mach number at that point. Equation (36.17) and Equation (36.18) are valid for any adiabatic flow whether thermodynamically reversible or not. They are, therefore, valid across a shock wave which is irreversible.

The ratio of the **stagnation pressure** to static pressure is obtained by substituting Equation (36.18) into the isentropic relation for pressure given by Equation (36.13) and letting state 2 be the stagnation state:

$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\gamma/(\gamma-1)} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma-1)} \tag{36.19}$$

The ratio of the **stagnation density** to static density is obtained by substituting Equation (36.18) into the isentropic relation for density given by Equation (36.13) and letting state 2 be the stagnation state:

$$\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{1/(\gamma-1)} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{1/(\gamma-1)} \tag{36.20}$$

Equation (36.19) and Equation (36.20) give the ratios of stagnation to static pressure and density, respectively, at a point in the flow field as a function of the Mach number at that point.

Equation (36.18), Equation (36.19), and Equation (36.20) provide important relations for stagnation properties and are usually tabulated as a function of Mach number M for $\gamma = 1.4$ (corresponds to air at standard conditions) in most standard compressible flow (gas dynamics) textbooks [Anderson, 2003; John, 1984; Zucrow and Hoffman, 1976]. Figure 36.2 was developed using ***COMPROP2*** [Tam et al., 2001] and shows the variation of these stagnation properties as a function of Mach number for $\gamma = 1.4$. It is important to note that the *local* value of stagnation property depends only upon the local value of the static property and the local Mach number and is independent of the flow process. Equation (36.18), Equation (36.19), and Equation (36.20) may be used to determine these local stagnation values, even for nonisentropic flow, assuming that the local static property and local Mach number are known. These equations also allow us to relate stagnation properties between any two points (say points 1 and 2) in the compressible flowfield. For example, if the actual flow between points 1 and 2 in the flowfield is reversible and adiabatic (isentropic), then T_0, p_0, and ρ_0 have constant values at every point in the

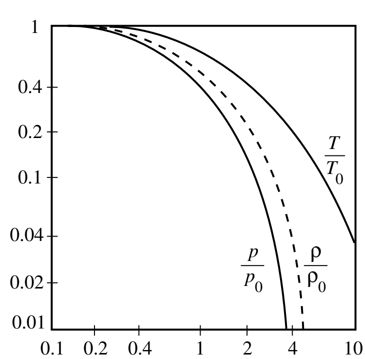

FIGURE 36.2 Stagnation property ratios for an ideal gas with $\gamma = 1.4$.

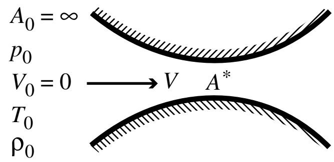

FIGURE 36.3 Converging–diverging nozzle.

flowfield. On the other hand, if the flow is irreversible and adiabatic, then only T_0 will remain constant at every point in the flow. However, for the case where the actual flow is irreversible and nonadiabatic, none of the stagnation properties stay constant between points 1 and 2 in the flowfield ($T_{01} \neq T_{02}$, $p_{01} \neq p_{02}$, and $\rho_{01} \neq \rho_{02}$).

36.7 Isentropic Flow with Area Changes

Nozzles are flow passages which accelerate the fluid to higher speeds. **Diffusers** accomplish the opposite, that is, they are used to decelerate the flow to lower speeds. These devices are quite common in gas turbines, rockets, and flow metering devices. In **incompressible flow** (ρ = constant), the volumetric flow rate (product of flow velocity, V, and the cross-sectional area, A) is constant. Thus, any passage which converges (causes A to decrease in the flow direction) is a nozzle, and any diverging passage (causes A to increase in the flow direction) is a diffuser. In fact, in any subsonic flow ($M < 1$), a converging channel accelerates and a diverging channel decelerates the flow. As will be shown in this section, just the opposite is true in supersonic flow ($M > 1$).

For the converging–diverging nozzle shown in Figure 36.3, the conservation of mass (continuity) under steady state conditions for this one-dimensional flow can be written as

$$\dot{m} = \rho V A = \text{constant} \tag{36.21}$$

The above equation can be used to relate the mass flow rate ($\dot{m}$) at different sections of the channel. Taking the logarithm of Equation (36.21) and then differentiating the resulting equation, we get

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0 \tag{36.22}$$

The differential form of the frictionless momentum equation for our steady one-dimensional flow is

$$dp + \rho V dV = 0 \tag{36.23}$$

Equation (36.23) could have also been obtained from the steady one-dimensional energy equation for an isentroic flow with no work interactions and no potential energy. Combining Equation (36.23) with Equation (36.22) and introducing the definition of Mach number, Equation (36.1), one can obtain

$$\frac{dA}{A} = -\frac{dV}{V}\left(1 - \frac{V^2}{dp/d\rho}\right) = -\frac{dV}{V}\left(1 - \frac{V^2}{a^2}\right)$$

or

$$\frac{dA}{dV} = \frac{A}{V}(M^2 - 1) \tag{36.24}$$

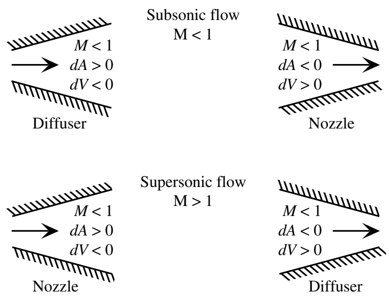

FIGURE 36.4 Area and velocity changes for subsonic and supersonic gas flow.

TABLE 36.1 Variation of Flow Properties in Converging or Diverging Channels

Type of Flow Passage	M	dA	dM	dV	dp	dT	dρ
Subsonic converging nozzle	<1	–	+	+	–	–	–
Subsonic diverging diffuser	<1	+	–	–	+	+	+
Supersonic converging diffuser	>1	–	–	–	+	+	+
Supersonic diverging nozzle	>1	+	+	+	–	–	–

where in the above equation for an isentropic flow, the speed of sound can be expressed as $a = \sqrt{dp/d\rho}$ [John, 1984]. Inspection of Equation (36.24), without actually solving it, shows a fascinating aspect of compressible flow. As mentioned at the beginning of this section, property changes are of opposite sign for subsonic and supersonic flow. This is because of the term $(M^2 - 1)$ in Equation (36.24). There are four combinations of area change and Mach number summarized in Figure 36.4. The variation of velocity, pressure, temperature, and density in converging–diverging channels in both subsonic and supersonic flow is tabulated in Table 36.1.

Equation (36.24) and Figure 36.4 have many ramifications. For M = 1 (sonic flow), Equation (36.24) yields dA/dV = 0. Mathematically this result suggests that the area associated with sonic flow (M = 1) is either a minimum or a maximum amount. The minimum in area is the only physically realistic solution. A convergent–divergent channel involves a minimum area (see Figure 36.3). These results indicate that the sonic condition (M = 1) can occur in a converging–diverging duct at the minimum area location, often referred to as the **throat** of a converging–diverging channel. Therefore, for the steady flow of an ideal gas to expand isentropically from subsonic to supersonic speeds, a convergent–divergent channel must be used. This is why rocket engines, in order to expand the exhaust gases to high-velocity, supersonic speeds, use a large bell-shaped exhaust nozzle. Conversely, for an ideal gas to compress isentropically from supersonic to subsonic speeds, it must also flow through a convergent–divergent channel, with a throat where M = 1 occurs.

The Mach number–area variation in a nozzle can be determined by combining the continuity relation [Equation (36.21)] with the ideal gas and isentropic flow relations. For this purpose, equate the mass flow rate at any section of the nozzle in Figure 36.3 to the mass flow rate under sonic conditions (at the throat, the flow is sonic and the conditions are denoted by an asterisk and are referred to as **critical conditions**):

$$\rho V A = \rho^* V^* A^*$$

or

$$\frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{V^*}{V} = \frac{\rho^*}{\rho}\frac{M^*}{M}\frac{a^*}{a} = \frac{1}{M}\left(\frac{\rho^*}{\rho_0}\right)\left(\frac{\rho_0}{\rho}\right)\sqrt{\frac{T^*/T_0}{T/T_0}} \tag{36.25}$$

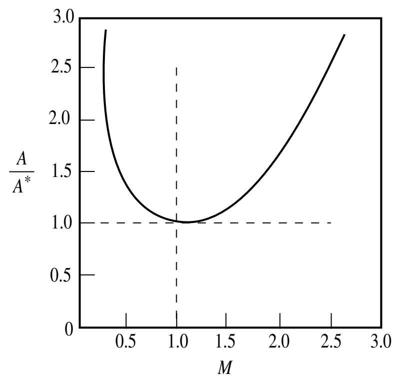

FIGURE 36.5 Variation of A/A^* with Mach number in isentropic flow for $\gamma = 1.4$.

where from Equation (36.2), $a = \sqrt{\gamma RT}$, and at the throat, the area of the throat is A^* and $M^* = 1 (V^* = a^* = \sqrt{\gamma RT^*})$, and for an isentropic flow ρ_0 and T_0 is constant throughout the flow. Substituting Equation (36.18) and Equation (36.20) into Equation (36.25) and recognizing that the **critical temperature ratio** (T^*/T_0) and the **critical density ratio** (ρ^*/ρ_0) is obtained from Equation (36.18) and Equation (36.20), respectively with M = 1, we obtain

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} \tag{36.26}$$

Equation (36.26) and ***COMPROP2*** [Tam et al., 2001] were used to generate the plot of A/A^* shown in Figure 36.5 for $\gamma = 1.4$. Numerical values of A/A^* versus M are also usually tabulated alongside stagnation properties given by Equation (36.18) through Equation (36.20) in most standard compressible flow textbooks, see, for example, Anderson [2003]. As can be seen from Figure 36.5, for each value of A/A^*, there are two possible isentropic solutions, one subsonic and the other supersonic. For example, from Equation (36.26) with $\gamma = 1.4$ or Figure 36.5, with $A/A^* = 2$, M = 0.3 and also M = 2.2. The minimum area (throat) occurs at M = 1. This agrees with the results of Equation (36.24), illustrated in Figure 36.4. That is, to accelerate a slow moving fluid to supersonic velocities, a converging–diverging nozzle is needed.

Under steady state conditions, the **mass flow rate** through a nozzle can be calculated from Equation (36.21) expressed in terms of M and the stagnation properties T_0 and ρ_0 from Equation (36.18) and Equation (36.19) as:

$$\dot{m} = \rho AV = \left(\frac{P}{RT}\right)A(M\sqrt{\gamma RT}) = PAM\sqrt{\frac{\gamma}{RT}} = \frac{\gamma P_0 AM}{\sqrt{\gamma RT_0}}\left(1+\frac{\gamma-1}{2}M^2\right)^{-\frac{(\gamma+1)}{2(\gamma-1)}} \tag{36.27}$$

Thus the mass flow rate of a particular fluid through a nozzle is a function of the stagnation properties of a fluid, the flow area, and the Mach number. The above relationship is valid at any location along the length of the nozzle.

For a specified flow area A and stagnation properties T_0 and ρ_0, the **maximum mass flow rate** through a nozzle can be determined by differentiating Equation (36.27) with respect to M and setting the result equal to zero. It yields M = 1. As discussed above, the only location in a nozzle where M = 1 is at the throat (minimum flow area). Therefore, the maximum possible mass flow passes through a nozzle when its throat is at the critical or sonic condition. The nozzle is then said to be **choked** and can carry no additional mass flow unless the throat is widened. If the throat is constricted further, the mass flow rate through the nozzle must decrease. We can obtain an expression for the maximum mass flow rate by substituting M = 1 in Equation (36.27):

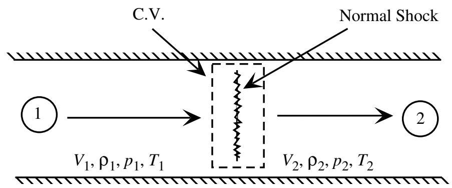

FIGURE 36.6 Stationary normal shock wave.

$$\dot{m}^* = \dot{m}_{max} = \gamma\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{P_0 A^*}{\sqrt{\gamma R T_0}} \tag{36.28}$$

Thus, for isentropic flow of a particular ideal gas through a nozzle, the maximum mass flow rate possible with a given throat area is fixed by the stagnation pressure and temperature of the inlet flow.

36.8 Normal and Oblique Shock Waves

When the flow velocity exceeds the speed of sound ($M > 1$), adjustments in the flow often take place through abrupt discontinuous surfaces called **shock waves**. This is one of the most interesting and unique phenomena that occurs in supersonic flow. A shock wave can be considered as a discontinuity in the properties of the flowfield. The process is irreversible. A shock wave is extremely thin, usually only a few molecular mean free paths thick (for air $\approx 10^{-5}$ cm). A shock wave is, in general, curved. However, many shock waves that occur in practical situations are straight, being either at right angles to the flow path (termed a **normal shock**) or at an angle to the flow path (termed an **oblique shock**). In case of a normal shock, the velocities both ahead (i.e., upstream) of the shock and after (i.e., downstream) the shock are at right angles to the shock wave. However, in the case of an oblique shock there is a change in the flow direction across the shock.

Attention will be given first to the changes that occur through a stationary normal shock. Fluid crossing a normal shock experiences a sudden increase in pressure, temperature, and density, accompanied by a sudden decrease in velocity from a supersonic flow to a subsonic flow. Consider an ideal gas flowing in a duct as shown in Figure 36.6. For steady state flow through a stationary normal shock, with no direction change, area change (shock is very thin), heat transfer (shock is adiabatic), or work done, the mass (continuity), momentum, and energy equations are:

Mass:
$$\rho_1 V_1 = \rho_2 V_2 \tag{36.29}$$

Momentum:
$$p_1 - p_2 = \rho_1 V_1 (V_2 - V_1) \tag{36.30}$$

Energy:
$$T_{01} = T_1 + \frac{V_1^2}{2c_p} = T_2 + \frac{V_2^2}{2c_p} = T_{02} \tag{36.31}$$

These equations, together with the definition of Mach number, Equation (36.1), the equation for speed of sound, Equation (36.2), the equation of state, Equation (36.3), and the equation for stagnation temperature, Equation (36.18), will yield two solutions. One solution, which is trivial, states that there is no change and hence no shock wave. The other solution, which corresponds to the change across a stationary normal shock wave, can be expressed in terms of the upstream Mach number:

$$M_2^2 = \frac{(\gamma-1)M_1^2+2}{2\gamma M_1^2-(\gamma-1)} \tag{36.32}$$

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2-(\gamma-1)}{(\gamma+1)} \tag{36.33}$$

$$\frac{p_{02}}{p_{01}} = \left[\frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}\right]^{\gamma/(\gamma-1)}\left[\frac{\gamma+1}{2\gamma M_1^2-(\gamma-1)}\right]^{1/(\gamma-1)} \tag{36.34}$$

$$\frac{T_2}{T_1} = [2+(\gamma-1)M_1^2]\frac{2\gamma M_1^2-(\gamma-1)}{(\gamma+1)^2 M_1^2} \tag{36.35}$$

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2} \tag{36.36}$$

The variations of p_2/p_1, ρ_2/ρ_1, T_2/T_1, p_{02}/p_{01}, and M_2 with M_1 as obtained from Equation (36.32) through Equation (36.36) and ***COMPROP2*** [Tam et al., 2001] are plotted in Figure 36.7 for $\gamma = 1.4$, and they are normally tabulated in standard compressible flow texts in "Normal Shock Tables," see, for example, Oosthuizen and Carscallen [1997]. From Figure 36.7 we can see that rather large losses of stagnation pressure occur across the normal shock. For an adiabatic process (flow across the normal shock is adiabatic but irreversible), the stagnation pressure represents a measure of available energy of the flow in a given state. A decrease in stagnation pressure, or increase in entropy ($s_2 > s_1$), represents an energy dissipation or **loss of available energy**. The increase in entropy across the normal shock can be related to the stagnation pressure ratio across the shock by substituting Equation (36.18) and Equation (36.19) in Equation (36.11) and recognizing that stagnation temperature remains constant across the normal shock, [see energy Equation (36.31)]. The result is:

$$s_2 - s_1 = -R\ln\frac{p_{02}}{p_{01}} \tag{36.37}$$

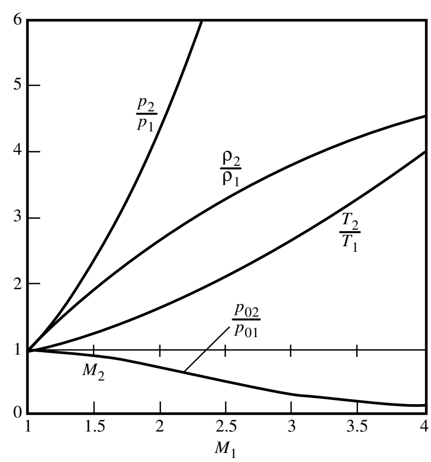

FIGURE 36.7 Variation of flow properties across a stationary normal shock wave for $\gamma = 1.4$.

From the second law of thermodynamics we must have $s_2 > s_1$. In order to minimize the loss of available energy across a normal shock, we need to have a small change in the stagnation pressure across the normal shock. Examination of Figure 36.7 shows that in order for this to happen the Mach number at the upstream of the normal shock (M_1) must be near unity.

If a plane shock is not perpendicular to the flow but inclined at an angle (termed an oblique shock), the shock will cause the fluid passing through it to change direction, in addition to increasing its pressure, temperature, and density, and decreasing its velocity. An oblique shock is illustrated in Figure 36.8; in this case the fluid flow is deflected through an angle δ, called the **deflection angle.** The angle θ shown in the figure is referred to as the **shock or wave angle** and the subscripts n and t indicate directions normal and tangent to the shock, respectively. The conservation of mass, momentum, and energy equations for the indicated control volume, see Figure 36.8, are:

Mass:
$$\rho_1 V_{1n} = \rho_2 V_{2n} \tag{36.38}$$

Momentum:
$$p_1 - p_2 = \rho_2 V_{2n}^2 - \rho_1 V_{1n}^2 \text{ (normal to the shock)} \tag{36.39}$$

$$0 = \rho_1 V_{1n}(V_{2t} - V_{1t}) \text{ or } V_{2t} = V_{1t} \text{ (tangent to the shock)} \tag{36.40}$$

Energy:
$$T_{01} = T_1 + \frac{V_{1n}^2}{2c_p} = T_2 + \frac{V_{2n}^2}{2c_p} = T_{02} \tag{36.41}$$

Since the conservation equations [Equation (36.38), Equation (36.39), and Equation (36.41)] for the oblique shock contain only the normal component of the velocity, they are identical to the normal shock conservation equations [Equation (36.29) through Equation (36.31)]. In other words, an oblique shock acts as a normal shock for the component normal to the shock, while the tangential velocity remains unchanged, [Equation (36.40)]. This fact permits the use of normal shock equations to calculate oblique shock parameters. To use normal shock equations for oblique shock calculations, in normal shock equations replace M_1 by M_{1n} and M_2 by M_{2n} where $M_{1n} = M_1 \sin\theta$ and $M_{2n} = M_2 \sin(\theta - \delta)$. Equations for M_{1n} and M_{2n} are dependent on δ and their values cannot be determined until the deflection angle is obtained. However, δ is a unique function of M_1 and θ. From the geometry of Figure 36.8 and some trigonometric manipulation, the following δ – θ – M relationship can be obtained:

$$\tan\delta = \frac{2\cot\theta(M_1^2 \sin^2\theta - 1)}{2 + M_1^2(\gamma + \cos 2\theta)} \tag{36.42}$$

Equation (36.42) specifies δ as a unique function of M_1 and θ. This relation is vital to the analysis of oblique shocks. The results obtained from Equation (36.42) are usually presented in the form of a graph as shown in Figure 36.9. Detailed oblique shock graphs (at times referred to as oblique shock charts) may be found in Oosthuizen and Carscallen [1997]. These charts along with the modified form of the normal shock equations are used for determination of the oblique shock properties. Figure 36.9 is a plot of shock angle versus deflection angle, with the upstream Mach number as a parameter. It is interesting

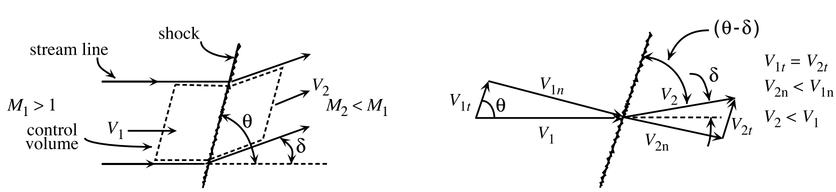

FIGURE 36.8 Oblique shock wave.

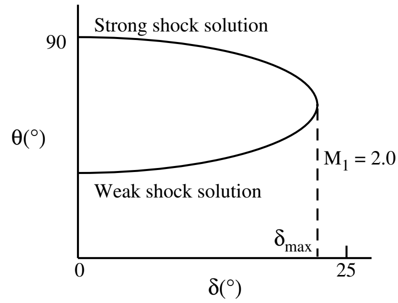

FIGURE 36.9 Oblique shock angle versus deflection angle and upstream Mach number for $\gamma = 1.4$.

to note that in the oblique shock figures, for a given initial Mach number (M_1) and a given deflection angle (δ), there are either two solutions (solid line and the dashed line in the figure) or none at all. Figure 36.9 shows that for a given M_1 (in this case $M_1 = 2.0$), a **maximum deflection angle** (δ_{max}) can be found (in this case $\delta_{max} = 23°$). This maximum varies from 0° at $M_1 = 1$ to about 45° as $M_1 \to \infty$. If δ_{max} is exceeded, the shock detaches, that is, moves ahead of the turning surface and becomes curved. For such a case, no solution exists on the oblique shock figures. In case of the attached shock, where two solutions are possible, the **weak shock** solution (solid line in Figure 36.9) is most common. A weak shock is the solution farthest removed from the normal shock case.

36.9 Rayleigh Flow

Ideal gas flow in a constant area duct with heating or cooling (stagnation temperature change) and without friction is referred to as **Rayleigh flow.** Heat can be added or removed to the gas by heat exchange through the duct walls, by radiative heat transfer, by combustion, or by evaporation and condensation. Although the frictionless (inviscid) assumption may appear unrealistic, Rayleigh flow is nevertheless useful for the analysis of jet engine combustors and flowing gaseous lasers. In these devices, the heat addition process dominates the viscous effects.

The conservation equations for the Rayleigh flow combined with the equation of state, the Mach number equation, and the definitions of the stagnation temperature and pressure, leads to the following equations; a complete analysis is given in Anderson [2003]:

$$\frac{T_0}{T_0{}^*} = \frac{(\gamma+1)M^2}{(1+\gamma M^2)^2}[2+(\gamma-1)M^2] \tag{36.43}$$

$$\frac{T}{T^*} = \frac{(1+\gamma)^2 M^2}{(1+\gamma M^2)^2} \tag{36.44}$$

$$\frac{p}{p^*} = \frac{(1+\gamma)}{(1+\gamma M^2)} \tag{36.45}$$

$$\frac{p_0}{p_0{}^*} = \left(\frac{1+\gamma}{1+\gamma M^2}\right)\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{\gamma/(\gamma-1)} \tag{36.46}$$

For convenience of calculation, in Equation (36.43) through Equation (36.46) sonic flow has been used as a reference condition, where the superscript asterisk (*) signifies properties at M = 1. In this way, the fluid properties for Rayleigh flow have been presented as a function of a single variable, the local Mach number. Equation (36.43) through Equation (36.46) and ***COMPROP2*** [Tam et al., 2001] were used to

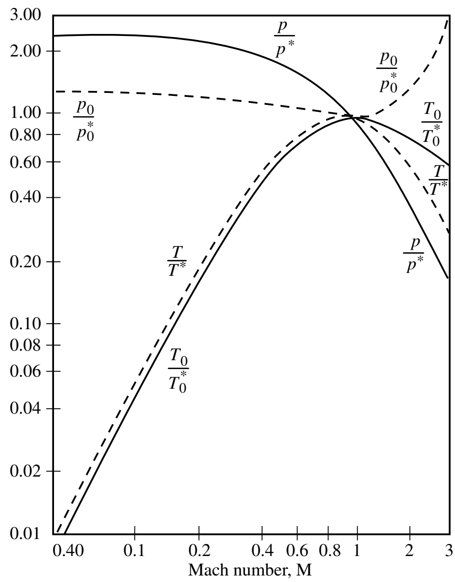

FIGURE 36.10 Rayleigh flow property variations with Mach number for $\gamma = 1.4$.

generate Figure 36.10 for $\gamma = 1.4$. These property variations are also normally tabulated in standard compressible flow texts in "Rayleigh Flow Tables," see, for example, John [1984]. Note that for a given flow no matter what the local flow properties are, the reference sonic conditions (the starred quantities) are constant values. Given particular inlet conditions to the duct (T_{01}, p_{01}, M_1) and $\dot{q}$ (the rate of heat transfer per unit mass of the flowing fluid), we can obtain the duct exit conditions after a given change in stagnation temperature (heating or cooling) as follows: The value of inlet Mach number (M_1) fixes the value of T_{01}/T_0^* from Equation (36.43) and thus the value of T_0^*, since we know T_{01}. The exit state T_{02}/T_0^* can then be determined from the following conservation of energy equation for this flow (for a given value of c_p or γ):

$$c_p(T_2 - T_1) + \frac{1}{2}(V_2^2 - V_1^2) = \dot{q} = c_p(T_{02} - T_{01}) = c_p \Delta T_0$$

or

$$\frac{T_{02}}{T_0{}^*} = \frac{T_{01}}{T_0{}^*} + \frac{\dot{q}}{c_p T_0{}^*} \tag{36.47}$$

By the value of T_{02}/T_0^*, then M_2, T/T^*, p_2/p^*, and p_{02}/p_0^* are all fixed from Equation (36.43) through Equation (36.46).

Referring to Figure 36.10 and comparing the flow properties with the T_0/T_0^* curve, several interesting facts about Rayleigh flow are evident. Considering the case of heating (increasing T_0/T_0^*), we notice that the increase in the stagnation temperature drives the Mach number toward unity for both subsonic and supersonic flow. After the Mach number has reached the sonic condition ($M = 1$), any further increase in heating (increase in stagnation temperature) is possible only if the initial conditions at the inlet of the duct are changed. Therefore, a maximum amount of heat can be added to flow in a duct, this maximum is determined by the attainment of Mach 1. Thus, flow in a duct can be chocked by heat addition. In this case flow is referred to as **thermally chocked.** Another interesting observation from Figure 36.10 is that the stagnation pressure always decreases for heat addition to the stream. Hence, combustion will cause a loss in stagnation pressure. Conversely, cooling tends to increase the stagnation pressure.

36.10 Fanno Flow

Flow of an ideal gas through a constant-area adiabatic duct with wall friction is referred to as **Fanno flow**. In effect, this is similar to a Moody-type pipe flow but with large changes in kinetic energy, enthalpy, and pressure in the flow.

The conservation equations for the Fanno flow combined with the definition of friction factor, equation of state, the Mach number equation, and the definitions of the stagnation temperature and pressure, leads to the following equations; a complete analysis is given in Anderson [2003]:

$$\frac{4f}{D}L^* = \left(\frac{1-M^2}{\gamma M^2}\right) + \frac{\gamma+1}{2\gamma}\ln\left[\frac{(\gamma+1)M^2}{2+(\gamma-1)M^2}\right] \tag{36.48}$$

$$\frac{T}{T^*} = \frac{(\gamma+1)}{2+(\gamma-1)M^2} \tag{36.49}$$

$$\frac{p}{p^*} = \frac{1}{M}\left[\frac{(\gamma+1)}{2+(\gamma-1)M^2}\right]^{1/2} \tag{36.50}$$

$$\frac{p_0}{p_0{}^*} = \frac{1}{M}\left[\frac{2+(\gamma-1)M^2}{(\gamma+1)}\right]^{(\gamma+1)/[2(\gamma-1)]} \tag{36.51}$$

Analogous to our discussion of Rayleigh flow, in Equation (36.48) through Equation (36.51) sonic flow ($M = 1$) has been used as a reference condition, where the flow properties are denoted by T^*, p^*, and $p_0{}^*$. L^* is defined as the length of the duct necessary to change the Mach number of the flow from M to unity and f is an average friction factor. Equations (36.48) through Equation (36.51) and ***COMPROP2*** [Tam et al., 2001] were used to generate Figure 36.11 for $\gamma = 1.4$. These equations are also normally tabulated in standard compressible flow texts in "Fanno Flow Tables," see, for example, John [1984].

Consider a duct of given cross-sectional area and variable length. If the inlet, mass flow rate, and average friction factor are fixed, there is a maximum length of the duct that can transmit the flow. Since the Mach number is unity at the duct exit in that case, the length is designated L^* and the flow is said to be **friction-chocked.** In other words, friction always derives the Mach number toward unity, decelerating a supersonic flow and accelerating a subsonic flow. From Equation (36.48) we can see that at any point in the duct (say point 1), the variable $f\,L^*/D$ depends only on the Mach number at that point (M_1) and γ. Since the diameter (D) is constant and f is assumed constant, then at some other point (say 2) a distance L ($L < L^*$) downstream from point 1, we have

$$\left(\frac{4f}{D}L^*\right)_2 = \left(\frac{4f}{D}L^*\right)_1 - \left(\frac{4f}{D}L\right) \tag{36.52}$$

From Equation (36.52), we can determine M_2. If in a given situation M_2 was fixed, then Equation (36.52) can be rearranged to determine the length of duct required (L) for Mach number M_1 to change to Mach number M_2.

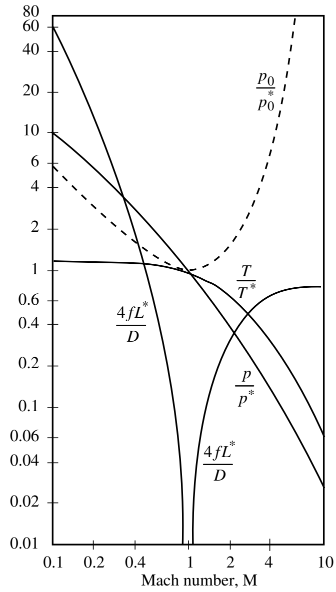

FIGURE 36.11 Fanno flow property variations with Mach number for $\gamma = 1.4$.

Defining Terms

Compressible flow — Flow in which the fluid density varies.

Isentropic flow — An adiabatic flow (no heat transfer) which is frictionless (ideal or reversible). For this flow the entropy is constant.

Stagnation state — A state that would be reached by a fluid if it were brought to rest isentropically (reversibly and adaibatically) and without work. The properties at the stagnation state are referred to as **stagnation properties** (or total properties).

Shock wave — A fully developed compression wave of large amplitude, across which density, pressure, temperature, and particle velocity change drastically. A shock wave is, in general, curved. However, many shock waves that occur in practical situations are straight, being either at right angles to the flow path (termed a **normal shock**) or at an angle to the flow path (termed an **oblique shock**).

Rayleigh flow — An idealized type of gas flow in which heat transfer may occur, satisfying the assumptions that the flow takes place in constant-area cross-section and is frictionless and steady, that the gas is ideal and has constant specific heat, that the composition of the gas does not change, and that there are no devices in the system that deliver or receive mechanical work.

Fanno flow — An ideal flow used to study the flow of fluids in long pipes; the flow obeys the same simplifying assumptions as Rayleigh flow except that the assumption there is no friction is replaced by the requirement the flow be adiabatic.

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