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PROCEEDINGS OF THE 1980 HEAT TRANSFER
AND FLUID MECHANICS INSTITUTE,
Melvin Gerstein and P. Roy Choudhury, Editors.
Stanford University Press, 1980.

LAMINAR FREE CONVECTION IN THE SUPERCRITICAL REGION WITH VARIABLE PROPERTIES

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Abstract - A technique was developed which allowed predictions of heat transfer to fluids in the supercritical region under variable property conditions. This technique proved to be more general, less complicated, easier to use, and more accurate in comparison to the existing prediction schemes. The heat transfer predictions are in good agreement with the existing variable property experimental data.

INTRODUCTION

Usually, rates of heat transfer by free convection are low compared to forced convection. However, several investigators [1][3] have observed high rates of free convection heat transfer in the critical and supercritical regions. This is primarily due to the fact that specific heat and compressibility for fluids in these regions increase significantly when compared to the variation of these properties in the subcritical region.

Convective heat transfer in the critical and supercritical region has applications in the use of hydrogen as a working fluid or fuel for both chemical and nuclear rockets and the use of water in electricity generating plants [1]. It is also important in the efficient cooling of turbine blades in high temperature gas turbines, in cooling of rocket motors by hydrocarbons, and in some refrigeration problems [2].

The growing list of applications necessitates the accurate analytical prediction of heat transfer in the supercritical region. The analysis is fairly difficult because of the peculiar and relatively large variation of the physical properties of fluids in the supercritical region with temperature. The variation of the specific heat at constant pressure with temperature is shown in Figure 1 for a pressure of 2.3×10^4 kPa (3400 psia).

In the present study the isothermal, vertical flat plate is chosen as the mathematical model and a steady, two-dimensional laminar boundary layer flow is assumed. Many have made analytical investigations of this problem, [3] through [7], but the existing prediction schemes are generally inefficient, complicated, and difficult to use. The methods are not general (i.e., they are dependent on the fluid of interest), and require extensive computer programming. They also require information on the development of analytical procedures for determining the thermodynamic and transport properties including their derivatives for the fluids of interest.

The purpose of this study, therefore, was to develop an accurate, efficient, and general scheme to predict heat transfer to fluids in the supercritical region under variable property conditions in laminar free convection on an isothermal vertical flat plate.

PHYSICAL MODEL

The physical model and the coordinate system of the present model are shown in Figure 2. Since it seems easier to visualize the case where the wall temperature exceeds the free stream temperature, the analysis will

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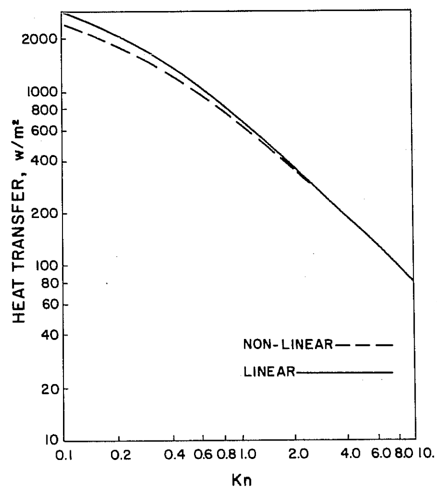

Figure 5. Effect of Knudsen Number on Heat Flux, T_{w2}/T_{w1} = 0.5, R_2/R_1 = 2.0.

be directed toward this situation. However, for the reverse case the results are also applicable.

FUNDAMENTAL EQUATIONS

The fundamental equations of conservation for steady laminar free convection in a boundary layer may be written in the following form, neglecting viscous dissipation:

Mass

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{1}$$

Momentum

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = g(\rho_\infty - \rho) + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{2}$$

Energy

$$\rho\, c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) \tag{3}$$

The driving force is the buoyancy term $g(\rho_\infty - \rho)$, formed by combining the pressure gradient $\partial p/\partial x$ with the body force ρg.

The boundary conditions appropriate to the problem are

$$\left.\begin{aligned} v &= 0 \\ u &= 0 \\ T &= T_w \end{aligned}\right\} y = 0 \qquad\qquad \left.\begin{aligned} u &= 0 \\ T &= T_\infty \end{aligned}\right\} y = \infty \tag{4}$$

REDUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

Equations (1) through (3) form a set of nonlinear, nonhomogeneous, simultaneous partial differential equations with variable coefficients, whose exact analytical solutions are unknown. These equations can be reduced to two nonlinear ordinary differential equations by means of the compressible stream function and a similarity variable. The stream function reduces the number of dependent variables as the conservation of mass equation is eliminated. The similarity variable transforms the partial differential equations into ordinary differential equations. The stream function and similarity variable are defined as follows:

Stream Function

$$u = \frac{\rho_w}{\rho}\left(\frac{\partial \psi}{\partial y}\right) \qquad\qquad v = -\frac{\rho_w}{\rho}\left(\frac{\partial \psi}{\partial x}\right) \tag{5}$$

Similarity Variable

$$\eta = C_w x^{-1/4} \int_0^y \frac{\rho}{\rho_w}\, dy \qquad \text{where} \qquad C_w = \left[\frac{g}{4\nu_w^2}\left(\frac{\rho_\infty - \rho_w}{\rho_w}\right)\right]^{1/4} \tag{6}$$

The new dependent variables F and θ are given by

$$F(\eta) = \left(\frac{1}{4\nu_w C_w}\right)\left(\frac{\psi}{x^{3/4}}\right) \tag{7}$$

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty} \tag{8}$$

The function F is related to the tangential and normal velocities u and v and the function θ is related to the temperature distribution.

Use of the transformation equations, Equations (5) through (8), leads to the two nonlinear ordinary differential equations of Sparrow and Gregg [8].

$$\frac{d}{d\eta}\left(\frac{\rho\mu}{\rho_w\mu_w} F''\right) + 3FF'' - 2(F')^2 + \frac{(\rho_\infty - \rho)\rho_w}{(\rho_\infty - \rho_w)\rho} = 0 \tag{9}$$

$$\frac{d}{d\eta}\left(\frac{\rho k}{\rho_w k_w}\theta'\right) + 3Pr_w\left(\frac{c_p}{c_{p_w}}\right)F\theta' = 0 \tag{10}$$

The boundary conditions, Equation (4), transform to

$$\left.\begin{array}{l} F = 0 \\ F' = 0 \\ \theta = 1 \end{array}\right\}\eta = 0 \qquad \left.\begin{array}{l} F' = 0 \\ \theta = 0 \end{array}\right\}\eta = \infty \tag{11}$$

The form of Equations (9) and (10) requires the physical properties to be expressed as functions of the independent variable η or as functions of the dependent variables F and θ. The procedures for solving these equations are long and tedious. However, at a particular pressure, the properties can be expressed as functions of temperature alone. Since temperature is a function of $\theta(\eta)$ (see Equation (8)), it is an easy matter to obtain the physical properties as functions of $\theta(\eta)$ once the equations for the physical properties are known. Manipulation of the coefficients in Equations (9) and (10) by the use of certain well-known thermodynamic relationships results in the following equations suggested by Parker and Mullin [5]. For detailed derivation see the thesis of Ghajar [9].

Momentum Equation

$$F''' = -(A1)\theta'F'' - 3(A2)FF'' + 2(A2)(F')^2 - (A3) \tag{14}$$

Energy Equation

$$\theta'' = -(B1)(\theta')^2 - 3(B2)F\theta' \tag{15}$$

Dimensionless Coefficients

$$A1 = \left[\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T}\right)_p + \frac{1}{\mu} \left(\frac{\partial \mu}{\partial T}\right)_p \right] (\Delta T) \qquad (16)$$

$$A2 = \frac{\rho_w \mu_w}{\rho \mu} \qquad (17)$$

$$A3 = (A2) \left[\frac{\rho_\infty - \rho}{\rho_\infty - \rho_w}\right] \frac{\rho_w}{\rho}$$

$$B1 = \left[\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T}\right)_p + \frac{1}{k} \left(\frac{\partial k}{\partial T}\right)_p \right] (\Delta T) \qquad (19)$$

$$B2 = (A2)\ (Pr) \qquad (20)$$

The five coefficients A1, A2, A3, B1, and B2 can be determined once the equations for the physical properties c_p, μ, k, and ρ are known as functions of temperature and pressure.

HEAT TRANSFER AT THE WALL

The local heat flux from the wall (plate) to the fluid may be calculated using Fourier's law:

$$\dot{q}''_{x,w} = -k_w \left(\frac{\partial T}{\partial y}\right)_{y=0} \qquad (21)$$

Introducing the dimensionless variables from Equations (6) through (8) in Equation (21), the local heat transfer rate in terms of the solutions of the differential equations becomes

$$\dot{q}''_{x,w} = -k_w (T_w - T_\infty)\ C_w x^{-1/4} \theta'(0) \qquad (22)$$

where $\theta'(0)$, the slope of the dimensionless temperature profile at the wall, is found from the solutions of Equations (14) and (15) along with the boundary conditions given by Equation (11).

PHYSICAL PROPERTIES

Accurate knowledge of both the thermodynamic and transport properties is necessary. That is, before the solutions to the reduced equations (14) and (15) can be obtained one must determine the coefficients expressed by Equations (16) through (20). These coefficients require the knowledge of the following thermodynamic and transport properties:

Thermodynamic Properties	Symbols
1. Density	ρ
2. Coefficient of volume expansion	$\frac{1}{\rho}(\partial\rho/\partial T)_p$
3. Specific heat at constant pressure	c_p

Transport Properties	Symbols
1. Absolute viscosity	μ
2. The term	$\frac{1}{\rho}(\partial\mu/\partial T)_p$
3. Thermal conductivity	k
4. The term	$\frac{1}{k}(\partial k/\partial T)_p$

The values of the above properties can be determined for any temperature at a particular pressure if expressions for the following are known for the fluid of interest:

1. An equation of state.
2. Specific heat at constant pressure as a function of temperature.
3. Absolute viscosity as a function of temperature.
4. Thermal conductivity as a function of temperature.

Some widely used working fluids in the supercritical region include the halocarbon compounds (Refrigerants), water, carbon dioxide, helium, hydrogen, nitrogen, oxygen, propane, and ammonia (see Ghajar [9]). Refrigerant-114 was the only working fluid for which all of the above information was available in the region where both pressures and temperatures were above critical. For many of the widely used fluids in the supercritical region, there is enough experimental thermodynamic and transport property data available in the literature (see Ghajar [9]), therefore, it was possible to develop the analytical procedures for determining the required expressions for the physical properties.

Determination of the four required expressions for physical properties to start the heat transfer calculations can be a very difficult task due to the peculiar variations of the physical properties of fluids in the supercritical region.

Once the types of equations that would represent the physical properties were recognized, it was possible to develop the analytical procedures necessary for determination of the physical property expressions based on the experimental data.

To demonstrate the universality of the present model, it was necessary to test the method with different fluids in the supercritical region.

Refrigerant-114, water, and carbon dioxide were chosen due to the availability of reliable physical property expressions and experimental data and because of the existence of theoretical and experimental heat transfer data (see Ghajar [9]).

The development of the physical property expressions and their derivatives is the subject of another publication. For more detailed information see Ghajar [9].

NUMERICAL METHOD FOR THE SOLUTION OF THE REDUCED EQUATIONS

The analytical investigations reported in the literature employ the classical fourth order Runge-Kutta numerical integration technique [2][3][5][6][7] to solve the set of Equations (14) and (15) and the boundary conditions, Equation (11). Konanur [2] in his work points out that the accuracy of the Runge-Kutta method depends upon the step size, $\Delta\eta$. Small values introduce round-off errors and large values give rise to truncation errors. Hence, for any problem the step size should be estimated by trial integrations.

Solution by the Runge-Katta method is possible if all the necessary boundary values are known. At the first boundary (η equal to zero) out of five required values (F, F', F'', θ, and θ') only three of them (F, F', and θ) are known. Hence, the other two unknown values (F'' and θ') have to be estimated. The solution of the problem is one of finding accurate values of these unknowns. Once the initial values are estimated, one can proceed with the integration process.

Since the unknown conditions $F''(0)$ and $\theta'(0)$ fix the shear stress and heat transfer at the plate, respectively, they must be determined very accurately. In the works reported in the literature, different iterative schemes were employed to overcome this difficulty. Fritsch et al. [3] and Konanur [2] first estimated values of $F''(0)$ and $\theta'(0)$ and then proceeded to a large value of η to check the results with the desired boundary conditions. Using an iterative technique, the estimated values were then adjusted until suitable agreement with the calculated values were noted. In order for their method to converge with reasonable computer time and accuracy, good estimates of the unknown boundary conditions are required. Konanur [2] also points out that the success of his model depends upon the case under study and upon how close the guessed values are to required values. Parker and Mullen [5] used the iterative method of Newton-Raphson to predict the unknown boundary conditions. Their method requires linearization of the reduced ordinary nonlinear differential equations and fairly good estimates for the starting boundary conditions.

In pursuit of the primary objectives of this study a numerical procedure which is easy to use, efficient, very accurate, and general, was sought. The numerical technique developed by Lentini and Pereyra [10] was used for this purpose. This is an adaptive finite difference method for first order nonlinear systems of ordinary differential equations subject to multipoint nonlinear boundary conditions. In this method variable order is provided through deferred corrections, while a built-in natural asymptotic estimator is used to automatically refine the mesh in order to achieve a required tolerance. This method is easy to use, extremely accurate, and it performs very efficiently in comparison with the results published in the literature for multipoint, nonlinear boundary value problems [10]. Moreover, the solutions of the reduced differential equations based on this method do not depend upon an initial estimate of the unknown boundary conditions. A thorough and complete discussion of this method is given in References [10] and [11]. The description and results of numerical solutions are given in [9].

HEAT TRANSFER RESULTS

In order to complete the analysis it was necessary to establish the validity and universality of the present variable property heat transfer model. This was accomplished by comparing the solutions of the present model with the former theoretical and experimental studies. Comparisons were made with the theoretical investigation of Parker and Mullin [5] for Refrigerant-114, experimental investigation of Fritsch and Grosh [12] for water, theoretical investigation of Nowak and Konanur [7] for water, and experimental investigations of Nishikawa et al. [13] for carbon dioxide.

Parker and Mullin [5] solved the problem for Refrigerant-114 at a pressure of 3.7×10^3 kPa (540 psia) and a temperature difference ($\Delta T = T_w - T_\infty$) of 2.8°C (5°F) for four values of the wall temperature, 152°C (305°F), 154°C (310°F), 157°C (315°F), and 160°C (320°F). Figure 3 shows that the heat transfer results of the present model are in excellent agreement with the theoretical results of Parker and Mullin. In this analysis although the physical property expressions of Parker and Mullin for Refrigerant-114 were used, the numerical methods for the solutions of the differential equations were completely different. The excellent agreement between the two models appears to be a good indication as to the validity of the present numerical technique. In comparison to the model of Parker and Mullin, the present model proved to be more general, less complicated, and easier to use.

Note in Figure 3 that the maximum value of local heat flux occurs at a wall temperature of 154°C (310°F), and a free stream temperature of 152°C (305°F). These temperatures are very close to the transposed critical temperature, T_M = 152°C (306°F), the temperature at which specific heat assumes its maximum.

Fritsch and Grosh [12] measured heat transfer in laminar free convection from a vertical flat plate, for water above and close to its critical point. The comparison of the present predicted heat transfer fluxes with the experimental data of Fritsch and Grosh, shown in Figure 4, is considered to be very good.

The lack of complete agreement between the present model and the experimental data of Fritsch and Grosh may, in part, be attributable to: (a) the fact that the Fritsch and Grosh experimental heat transfer data was obtained from a surface more closely resembling constant heat flux conditions than one at a uniform temperature. A constant heat flux surface gives higher heat fluxes than those from a corresponding isothermal heat transfer surface; (b) the fact that the 1.27 cm (1/2 in.) high vertical heating wall was so small that Rayleigh numbers in their experiments were sufficiently small, but their results could not be said to be an experiment for a vertical wall;

and (c) the errors in the physical properties of water in the supercritical region.

Figure 4 also compares the present model with the theoretical model of Nowak and Konanur [7]. The present model predicts the experimental data consistently better than the analytical model of Nowak and Konanur with the superiority of the present model more significant for the cases where the difference between the wall and the free stream temperatures become large.

Nishikawa et al. [13] measured heat transfer data in laminar free convection from a vertical flat plate for carbon dioxide close to its critical point. In Figure 5 the predictions of the present model show excellent agreement with their experimental data.

Figure 5 shows that the present model predicts the experimental data of Nishikawa et al. [13] for carbon dioxide far better than the analytical model of Nishikawa and Ito [6]. The superiority of the present model becomes more pronounced for the cases where the difference between the wall and the free stream temperatures become large.

In general, the present prediction model appears to be in very good agreement with the available experimental data of fluids in the supercritical region. Aside from the specific points that were mentioned in the discussion of the results, in general the lack of complete agreement between the existing model and the experimental data might be attributable to (a) errors in the thermodynamic and transport properties of fluids in the supercritical region, and to (b) the deficiencies in the existing experimental heat transfer data. Among many other items which must be examined, the technique to measure the temperature of the heating wall were not suitable in some experiments, and pressure vessels in some apparatus were too small to realize the condition of stagnant fluid in infinitely large spaces. The present model, in all cases compared, proved to be better than the existing theoretical models in predicting the experimental heat transfer data.

The superiority of the present model was far more pronounced for cases where the difference between the wall and the free stream temperatures become large. In contrast to the theoretical models of Nowak and Konanur [7] for water, and Nishikawa and Ito [6] for carbon dioxide, the present model does not utilize in a direct manner the experimental values of the physical properties of fluids in the supercritical region and their derivatives for the problem under study. The present model also proved to be more general (i.e., it is independent of the fluid of interest), less complicated, easier to use, and more accurate in comparison to the specific models of Nowak and Konanur [7] for water, Nishikawa and Ito [6] for carbon dioxide, and Parker and Mullin [5] for Refrigerant-114.

Figure 6 compares the theoretical values of local heat fluxes predicted by the present model with the experimental values of supercritical water and carbon dioxide. The correlation between the theoretical and experimental values is reasonable and all of the supercritical free convection heat transfer data are predicted by the theory within an error of $\pm$ 20 percent. This shows that the present model is reasonably good at predicting heat transfer to fluids in the supercritical region under variable property conditions in laminar free convection on a vertical flat plate.

Nomenclature

c_p = specific heat at constant pressure

C_w = defined by Equation (6)

$F(\eta)$ = similarity variable, defined by Equation (7)

g = acceleration due to gravity

k = thermal conductivity

p = pressure

Pr = Prandtl number

$\dot{q}''$ = heat flux

T = temperature

T_M = pseudocritical or transposed critical temperature

u = velocity component along the plate

v = velocity component perpendicular to the plate

x = coordinate along the plate

y = coordinate perpendicular to the plate

Greek symbols

ΔT = temperature difference, $\Delta T = T_w - T_\infty$

δ = boundary layer thickness

η = similarity coordinate as defined by Equation (6)

μ = absolute viscosity

ν = kinematic viscosity, $\nu = \mu/\rho$

ρ = density

ψ = stream function, defined by Equation (5)

$\theta(\eta)$ = dimensionless temperature, defined by Equation (8)

Subscripts

c = refers to critical property

w = evaluated at the wall temperature

x = evaluated at a particular point along a surface, a local parameter as opposed to a mean parameter

∞ = evaluated at the free stream temperature

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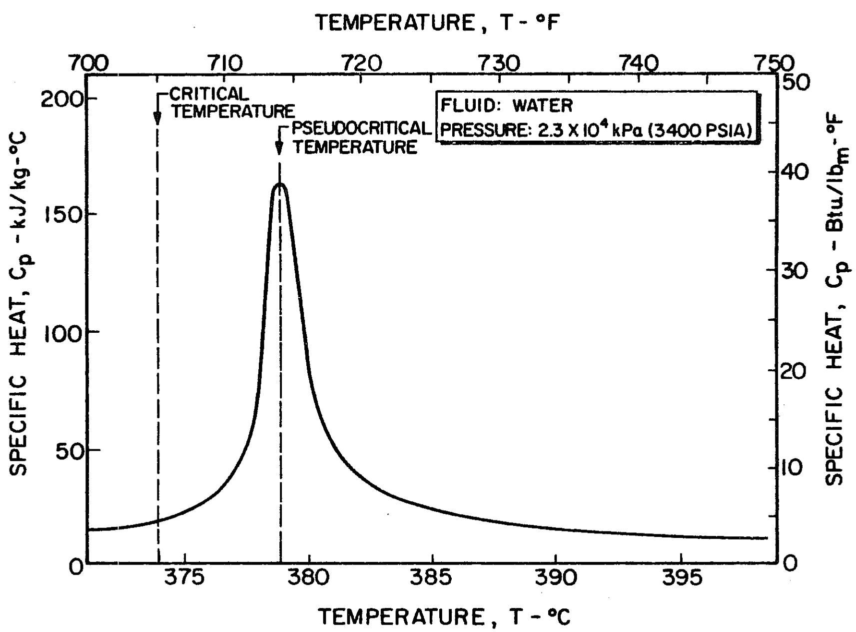

FIG. 1. Specific Heat in the Supercritical Region

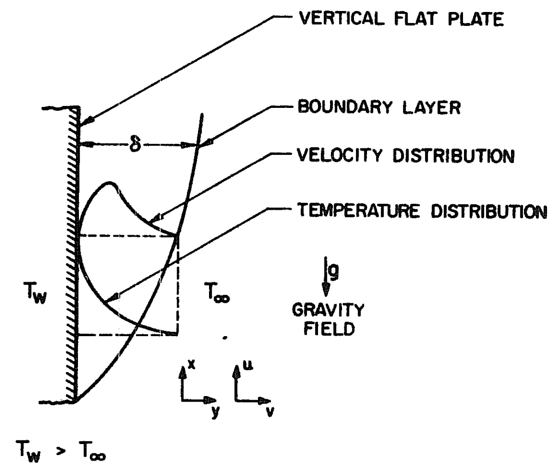

FIG. 2. Physical Model and Coordinate System

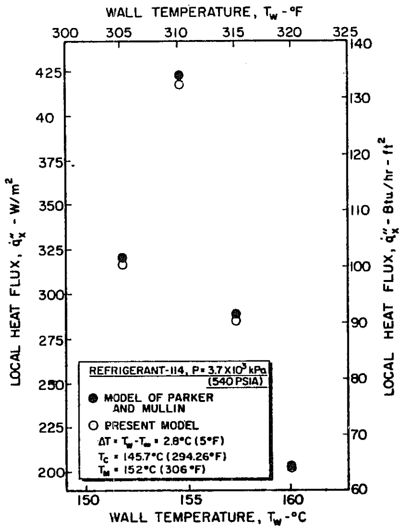

FIG. 3. Comparison of Predicted Values of Local Heat Flux with Model of Parker and Mullin [5]

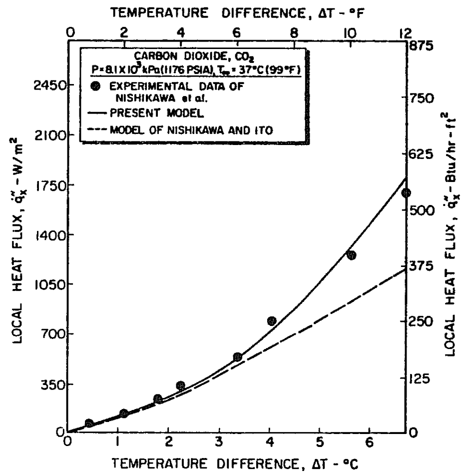

FIG. 5. Comparison of Present Model With Experimental Data of Nishikawa et al. [13] and Model of Nishikawa and Ito [6]

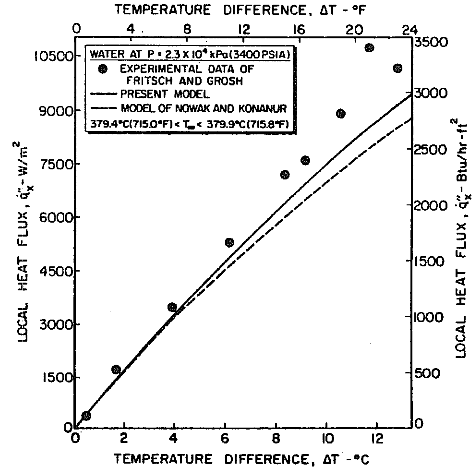

FIG. 4. Comparison of Present Model With Experimental Data of Fritsch and Grosh [12] and Model of Nowak and Konanur [7]

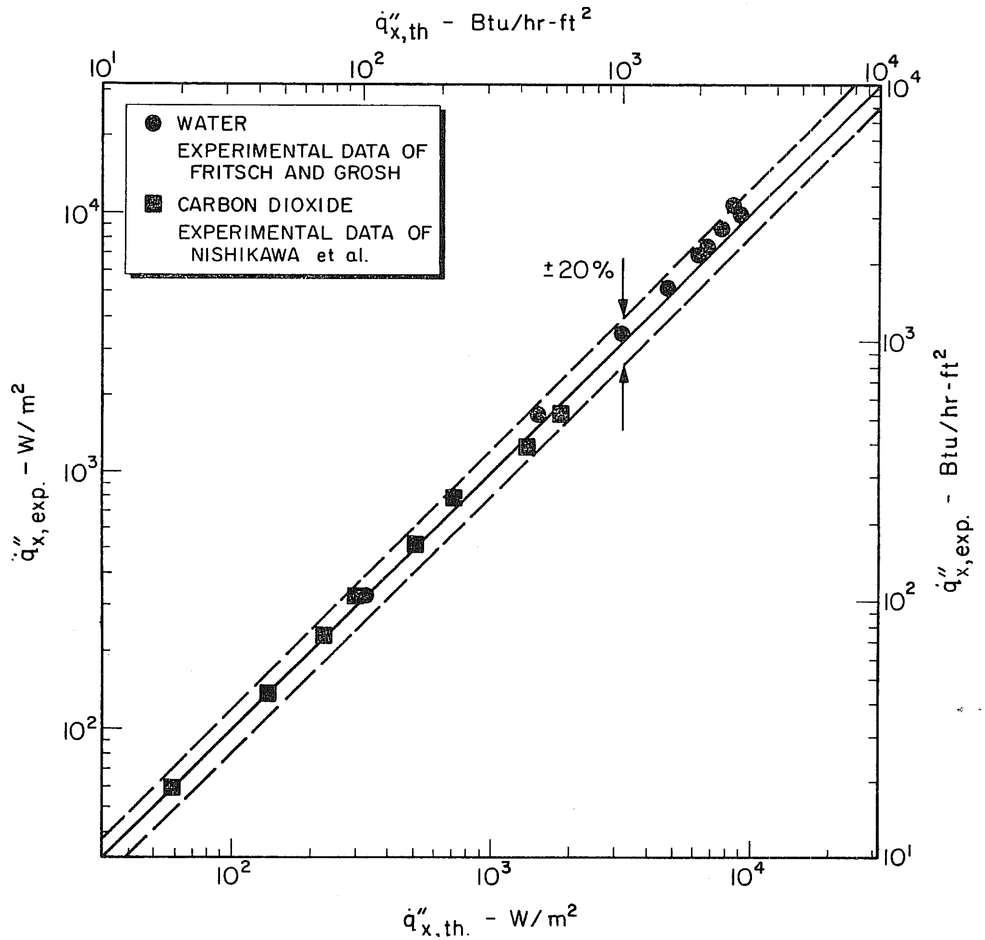

FIG. 6. Comparison of Predicted Local Heat Flux Using the Present Model With Measurements