INT. COMM. HEAT MASS TRANSFER 0735-1933/87 $3.00 + .00
Vol. 14, pp. 67-79, 1987 ©Pergamon Journals Ltd. Printed in the United States

NUMERICAL SOLUTIONS OF HEAT CONDUCTION AND SIMPLE FLUID FLOW PROBLEMS

H. Th. Jakobsson*, Graduate Student and **A. J. Ghajar**, Associate Professor
School of Mechanical and Aerospace Engineering
Oklahoma State University
Stillwater, Oklahoma 74078

(Communicated by J.P. Hartnett and W.J. Minkowycz)

ABSTRACT

This paper tests the accuracy and versatility of a finite difference control volume based computer code for numerical solutions of heat conduction (steady and unsteady), and simple fully developed laminar duct flow problems. The program's range of applicability and special features are discussed and its accuracy is tested on ten engineering problems. The rate of convergence seems to be very high and the accuracy excellent when compared to analytical and other numerical solutions.

Introduction

With the advent of the computers, there has been a steady growth of interest in computer aided numerical calculations. This interest has spread all over the technical field and heat transfer and fluid flow predictions are no exceptions. Due to the enormous possibilities of prediction methods in this field, many computer codes have been developed which are applicable to a variety of practical engineering problems. One such code was developed by Patankar and Baliga [1], based on the procedures and methodology outlined by Patankar [2].

*Present Address: National Energy Authority, Geothermal Engineering Division, Reykjavik, Iceland.

This computer code is based on a control volume finite difference method and is designed to solve steady or transient, one and two dimensional problems in heat conduction and simple fully developed laminar duct flows.

The purpose of this paper is to apply this program to several engineering problems and to compare the results to exact analytical solutions and other numerical solutions in the open literature which include finite difference methods, the boundary integral equation method, and finite element methods.

The program which was originally written for a CDC computer was modified so that it can be used on an IBM 3081 computer, and the problems shown here were run in double precision. An extensive user guide has been developed to make the program easier to use [3]. The computer code hereafter will be referred to as **HEFFLO** (Heat and Fluid Flow).

Program Capabilities

Range of Applicability

The main purpose of the computer code is to solve problems that obey a general differential equation, which can be written in a Cartesian tensor form as:

$$\frac{\partial}{\partial x_i}\left(\Gamma \frac{\partial \phi}{\partial x_i}\right) + S = \frac{\partial}{\partial t}(\rho\ c\ \phi) \qquad (1)$$

where ϕ is the general dependent variable to be found, x_i are the coordinates, Γ is the general diffusion coefficient, ρ is the density, c is the specific heat at a constant pressure and S is the source term.

For heat conduction problems, Γ is equal to the thermal conductivity k and ϕ stands for temperature T. Now Eq. (1) becomes:

$$\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + S = \frac{\partial}{\partial t}(\rho\ c\ T) \qquad (2)$$

Equation (1) is also the governing equation for simple fully developed laminar flow. Simple fully developed flow is defined as a flow with no cross-stream velocities, i.e., v = 0, w = 0, and u = u(y,z). For this flow the pressure P is constant over each cross section and linear with x. For this case, Eq. (1) simplifies to:

$$\mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) - \frac{\partial P}{\partial x} = 0 \qquad (3)$$

where μ is the absolute viscosity.

For the same type of flow, the temperature profile, T, can also be calculated. Here Eq. (1) reduces to:

$$\frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) - \rho\ c\ u \frac{\partial T}{\partial x} = 0 \qquad (4)$$

The last term in Eqs. (3) and (4) are handled through the source term S.

Special Features

The discretization equations used in the program are developed by a finite difference **control volume method** where the grid points are located at the geometrical center of each control volume. Since the control volume approach is based on an energy balance, it is necessary for heat conduction problems that the choice of the interface conductivity leads to a good representation for the heat flux at the interfaces. For this purpose the program uses the **harmonic-mean** formula for interface conductivity calculations. This can handle severe changes in the distribution of the conductivity between control volumes, as well as uniform conductivity. Boundary fluxes are perhaps the most important outcome of the computation. To obtain better accuracy for the boundary fluxes, the program uses a **special higher-order boundary flux formulation.**

For the transient cases the program employs the **fully-implicit method** where the value at each point is a function of the values of the neighboring points. Therefore, **iteration** must take place for each time step. By using the fully-implicit method oscillations in the numerical solution are eliminated and the discretization equations are simplified. In solving the discretization equations for each iterative sweep the program employs the **line by line TDMA** (Tri-Diagonal Matrix Algorithm) method and the rate of convergence is improved by a special **block-correction procedure** proposed by Settari and Aziz [4].

The **over- or under-relaxation** method is a built-in optional feature in the program. The code is also equipped to handle **three coordinate systems,** Cartesian, polar and axi-symmetric, with **uniform or nonuniform grid systems.** The main attractiveness of HEFFLO is its versatility. The program is divided into two parts, an invariant part and a problem dependent part. Therefore, the user does not have to go through extensive work of rewriting the program every time, but can write a relatively short subroutine for each problem. Further details about the program may be found in refs. [1-3].

General Description of the Computer Code

The computer code consists of eleven executable subroutines, five in **MAIN** (SOLVE, SETUP1, SETUP2, UGRID and PRINT) and six in **USER** (GRID, START, DENSE, OUTPUT, GAMSOR and TPRINT). Also there are some preliminaries which includes a special **BLOCK DATA** subprogram in MAIN. This subprogram consists of default values, some of which are redefined in USER. The MAIN part of HEFFLO is fixed and there should not be any need to change it unless specifically desired. The USER is the problem dependent part of HEFFLO and needs to be rewritten in each case. A flow chart of the program is given in Figure 1. The program works in the following way:

First **GRID** is called. This subroutine is in the USER. The purpose of GRID is mainly to construct the grid system for the problem involved. This can be done in two ways. If the user wants a uniform grid, a special subroutine, **UGRID**, is available. If a nonuniform grid is preferred the user must construct it in GRID. Also this is a good place to define some problem related quantities to be used later on in the program. These are for example thermal conductivities, heat transfer coefficients, boundary temperatures, etc. Finally the default values from the BLOCK DATA subroutine can be redefined in GRID.

Next **SETUP1** is called. This subroutine is in the MAIN. Here some once-and-for-all calculations concerning the grid system are made. These include the distances to each grid point, the distances between the points, and width of the control volumes. Finally some arrays are initiated in SETUP1.

Now **START** is called. START is in the USER and the only purpose of this subroutine is to give starting values for the iterations and specify those boundary conditions that are fixed.

Now the iterataion loop starts.

Next **DENSE** is called. DENSE is in the USER and is only necessary in transient cases. In these cases DENSE provides the values of the coefficients ρc in Eq. (1) for all the internal points.

Next, subroutine **OUTPUT** is called. OUTPUT is in USER. Here the user can calculate and print out intermediate results that are of importance. These, for example can be various heat fluxes (for heat conduction problems) or friction factor, Reynolds, and Nusselt numbers (for flow problems). If a convergence criterion is desired, OUTPUT is the right place to establish this. To get a printout of the converging field, the user can call subroutine **PRINT** which is in MAIN. To get a title for the field printout, PRINT calls **TPRINT** which is in USER.

Now **SETUP2,** which is in **MAIN,** is called. This is really the heart of the program. First SETUP2 calls **GAMSOR** which is in USER. In GAMSOR the diffusion coefficient Γ in Eq. (1) is defined for all internal points. To remind the reader, Γ usually either stands for the thermal conductivity, k or the absolute viscosity, μ. In GAMSOR the source terms are also defined for all the points. It should be noted that the source terms are also used to handle those boundary conditions that are not fixed.

With the information already gathered from the USER subroutines, SETUP2 can now calculate all the discretization coefficients.

Next SETUP2 calls subroutine **SOLVE** which is in MAIN. In SOLVE the discretization equations are solved by using the line by line iteration method and the block correction method. This ends one iteration. The program continues in this manner until either the convergence criterion or the last iteration has been reached.

The above discussion should give the reader a good insight into how the computer code HEFFLO works. Further details about the program may be found in refs. [1-3].

Testing The Computer Code

To determine the accuracy and the versatility of the computer code, the code was tested on ten different engineering problems of varying degrees of complexity. These problems where chosen so as to test the full capabilities of the code. For the purpose of this paper, only three of these test problems will be presented here. Details about other test problems may be found in Jakobsson [3] and they include the following problems:

1. **Steady Conduction in a Plane (various boundary conditions)**
   The numerical solutions were compared with the analytical solutions given in Lilley [5].

2. **Steady Conduction in an Infinite Cylinder**

   The results of the numerical solution were compared with the analytical solution given in Schneider [6].

3. **Steady Conduction in an Axi-symmetric Cylinder**

   The numerical results were compared with the analytical solution given in Schneider [6].

4. **Fully Developed Flow in a Circular Duct**

   The numerical solutions were compared with the analytical solutions given in Kays and Crawford [7] for velocity and temperature profiles.

5. **Transient Heat Conduction in a Slab**

   The numerical results were compared with the analytical and finite element solutions of Orivuori [8].

6. **Heat Conduction in an Infinite Hollow Cylinder**

   The results of the numerical solution were compared with analytical solution [9] and a four node linear finite element solution [10].

Test Problem 1 - Various Boundary Conditions for Heat Conduction in a Slab

In this problem HEFFLO's solution is compared to both an analytical solution and a solution obtained by a boundary integral equation method (BIE) [11]. The problem's geometry, which is really a set of seven problems, is shown in Figure 2. The boundary conditions are different for each of the seven cases considered, but because the problems are all one dimensional, an analytical solution (at least for the boundary points) can be obtained by a simple energy balance.

In the comparisons, two things were considered. Firstly, the temperature of the end points were compared to both the analytical solution and the BIE solution [11], and secondly a comparison of the amount of computer time needed for both of the programs (HEFFLO and the BIE program) was made. For the BIE method the exact definition of the reported computational time is not clear. It is assumed that the stated computational time is the sum of the compilation time and the execution time. The results are shown in Table 1. Based on this table, the programs seem to be equally economical, however the accuracy of HEFFLO is much higher.

Test Problem 2 - Steady Conduction in a Plane

Figure 3 shows the geometry for problem 2. The analytical solution for this problem can be shown to be [12]:

$$T^* = \alpha - \zeta + \frac{1}{Bi} + \sum_{n=1}^{\infty} \frac{2}{n\pi} [\frac{b}{Q} \int_0^{\varepsilon} q(\xi)\cos(n\pi\xi)\, d\xi] \times \cos(n\pi\xi)\, [\psi_n \cosh(n\pi\xi) - \sinh(n\pi\xi)]$$

where ψ_n is defined by

$$\psi_n = \frac{n\pi \cosh(n\pi\alpha) + Bi \sinh(n\pi\alpha)}{n\pi \sinh(n\pi\alpha) + Bi \cosh(n\pi\alpha)}$$

and

$$T^* = k(T - T_f)/Q$$

$$Q = \int_0^a q(x)\, dx$$

$$Bi = hb/k$$

$$\alpha = c/b$$

$$\varepsilon = a/b$$

$$\xi = x/b$$

$$\zeta = y/b$$

The grid system used in running this problem was a 22 x 22 uniform grid. HEFFLO's solution is compared to three other solutions. Analytical solution [12], numerical solution based on Galerkin weighted residual formulation [13] and a numerical method based on a special control volume finite element formulation [13].

The results are shown in Figure 4. This figure clearly shows that HEFFLO's solution is the best one for the higher values of x, but not very good for the lower values of x. The accuracy of HEFFLO could probably be increased by using a finer grid system.

Test Problem 3 - Fully Developed Flow in a Rectangular Duct

The problem here involves finding the Nusselt number (Nu) and the friction factor Reynolds number product ($C_f Re$) for a simple fully developed flow in a rectangular duct. The geometry is shown in Figure 5. Three boundary conditions are considered, one in some detail. These are:

Case 1: Constant heat flux in the axial direction and constant peripheral wall temperature at each cross section.

Case 2: Constant heat flux in the axial and peripheral directions.

Case 3: Constant wall temperature in the axial and peripheral directions. These cases will now be examined one by one. The solutions are compared to both numerical and analytical solutions presented by Shah and London [14].

Case 1

For this case, eight different problems were solved and the comparison with analytical and other numerical solutions is shown in Table 2. It should be noted that the solutions presented for the case where none of the sides were insulated were obtained analytically [14]. All the other solutions were obtained numerically by Shah and London [15] by using a finite difference technique. Based on the results of Table 2, the computer code predicts the results very well.

Case 2

Here only one case was considered, $\alpha = 1$ and none of the sides adiabatic. The Nusselt number obtained by HEFFLO is Nu = 3.100 compared to Nu = 3.091 from refs. [14, 15]. This is a difference of 0.29%.

Case 3

Here only one case was considered, $\alpha = 1$ and none of the sides adiabatic. The Nusselt number obtained by HEFFLO is Nu = 2.980 compared to Nu = 2.976 from refs. [14, 15]. This is a difference of 0.13%.

Conclusions

The accuracy and versatility of this computer code which is based on a special finite difference control volume formulation was tested on ten engineering problems. These test problems demonstrated that this code can handle the three mentioned coordinate systems (Cartesian, polar and axi-symmetric), uniform and non-uniform thermal conductivity and thermal capacity, transient and non-transient problems, and a variety of boundary conditions (convection, radiation, etc.).

In general HEFFLO's (Heat and Fluid Flow) solutions were more accurate than the other numerical methods to which it was compared, and equally economical. From these results it can be safely stated that this computer code is compatible with the other existing numerical methods for solving heat conduction and simple fully developed laminar duct flow problems, and highly recommended because of its simplicity and accuracy.

References

1. S. V. Patankar, and B. R. Baliga, Computation of Heat Transfer and Fluid Flow, Course Notes - A University of Minnesota Short Course for Engineers and Scientists, Minneapolis, Minnesota, Nov. 16-19 (1981).

2. S. V. Patankar, Numerical Heat Transfer and Fluid Flow, McGraw-Hill, New York (1980).

3. H. Th. Jakobsson, Control Volume Based Finite Difference Formulation for Heat Conduction and Simple Fluid Flow Problems, M.S. Report, School of Mechanical and Aerospace Engineering, Oklahoma State University, Stillwater, Oklahoma (1982).

4. A. Settari and K. Aziz, SIAM J. Numer. Anal. 10, 506 (1973).

5. D. G. Lilley, Conduction Heat Transfer Course Notes for MAE 5843, Oklahoma State University, Stillwater, Oklahoma (1981).

6. P. J. Schneider, Conduction Heat Transfer, Addison-Wesley, Mass. (1955).

7. W. M. Kays, and M. E. Crawford, Convective Heat and Mass Transfer, 2nd ed. McGraw-Hill, New York (1980).

8. S. Orivuori, Int. J. for Numerical Methods in Engineering 14, 1461 (1979).

9. H. S. Carslaw, and J. C. Jaeger, Conduction of Heat in Solids, 2nd edn. Oxford University Press, London (1948).

10. A. J. Baker, Finite Element Analysis in Fluid Dynamics and Heat Transfer, Seventh Short Course Notes, Vol. II, held at the University of Texas, Austin, Texas, July 12-16 (1982).

11. M. S. Khader, and M. C. Hanna, ASME J. of Heat Transfer 103, 26 (1981).

12. G. E. Schneider, M. M. Yovanovich, and R. L. D. Cane, AIAA Journal of Spacecraft and Rockets 17, 372 (1980).

13. G. E. Schneider, and M. Zedan, Control Volume Based Finite Element Formulation of the Heat Conduction Equation, AIAA/ASME 3rd Joint Thermophysics Fluids, Plasma and Heat Transfer Conference, St. Louis, Missouri, June 7-11 (1982).

14. R. K. Shah, and A. L. London, Laminar Flow Forced Convection in Ducts, Advances in Heat Transfer, Supplement 1, Academic Press, New York (1978).

15. R. K. Shah, and A. L. London, Laminar Flow Forced Convective Heat Transfer in Straight and Curved Ducts - A Summary of Analytical Solution, TR No. 75, Dept. Mech. Engr., Stanford University, Stanford, California (1971).

TABLE 1
Comparison of Exact, BIE, and HEFFLO Solutions for Test Problem 1

Case	Boundary Conditions B1	Boundary Conditions B2	a	b	k	h	ε	SF	Exact	BIE	%Error	HEFFLO	%Error	Computational time (sec) BIE	Computational time (sec) HEFFLO***
1	Const. T T_1 = 200°C	Conv. $T_{\infty 2}$ = 50°C	2	3	1.5	10	-	-	T_2 = 57.14°C	57.11°C	-0.053	57.14°C	0.000	2.11	2.90 + .52 = 3.42
2	Const. q q = 1000 W/m^2	Rad. $T_{\infty 2}$ = 350°K	1	0.5	2.0	-	1	1	T_1 = 675.07 T_2 = 425.07 [°K]	674.87 425.00 [°K]	-0.030 -0.016	675.09 425.09 [°K]	0.003 0.005	3.77	2.93 + .52 = 3.45
3	Conv. $T_{\infty 1}$ = 600°K	Rad. $T_{\infty 2}$ = 300°K	10	10	5.0	20	1	1	T_1 = 593.18 T_2 = 320.17 [°K]	593.17 320.62 [°K]	-0.002 0.141	593.17 320.17 [°K]	-0.002 0.000	4.48	2.99 + .52 = 3.51
4	Const. T T_1 = 300°K	Conv. + Rad. $T_{\infty 2}$ = 500°K	1	0.25	0.2	1	1	1	T_2 = 494.61 [°K]	494.65 [°K]	0.008	494.61 [°K]	0.000	2.83	3.01 + .54 = 3.55
5	Const. T T_1 = 100°C	Conv. $T_{\infty 2}$ = 20°C	1	2	k_5*	3	-	-	T_2 = 29.66 [°C]	29.61 [°C]	-0.169	29.66 [°C]	0.000	3.14	2.96 + .53 = 3.49
6	Const. q q = 50W/m^2	Conv. $T_{\infty 2}$ = 25°C	1	4	0.5	h_6**	-	-	T_1 = 435.99 T_2 = 35.99 [°C]	434.91 35.96 [°C]	-0.248 -0.083	435.99 35.99 [°C]	0.000 0.000	3.72	2.91 + .60 = 3.51
7	Const. T T_1 = 2 x 10^4°C	Rad. $T_{\infty 2}$ = 0°C	2	2	1.0	-	1	1	T_2 = 642.88 [°C]	641.88 [°C]	-0.156	642.80 [°C]	-0.012	30.48	2.94 + .55 = 3.49

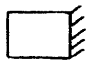
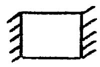
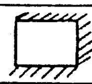
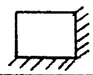
* $k_5 = 0.5\ (1 + 0.01T)$

** $h_6 = 2.5\ (T_2 - T_{\infty 2})^{1/4}$

*** The numbers are the compilation and execution times respectively.

k = Thermal conductivity [W/m°K]
h = Heat transfer coefficient [W/m^2°K]
ε = Emissivity (for radiation)
SF = Shape factor (for radiation)

TABLE 2
Comparison of Analytical and Numerical Results
for Test Problem 3, Case 1

		Nusselt Number					
α							C_f Re
1	[14]	3.608	3.556	4.094	2.712	2.836	14.23
	HEFFLO	3.629	3.583	4.104	2.687	2.851	14.21
	% Difference	0.58	0.76	0.24	-0.92	0.53	-0.12
0.2	[14]	5.738	-	-	-	-	19.07
	HEFFLO	5.808	-	-	-	-	18.96
	% Difference	1.22	-	-	-	-	-0.58
2	[14]	4.123	-	-	-	-	15.55
	HEFFLO	4.151	-	-	-	-	15.51
	% Difference	0.68	-	-	-	-	-0.24

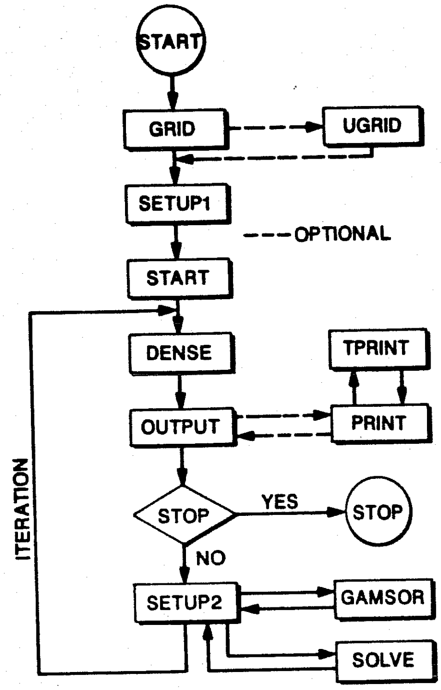


FIG. 1
Flow diagram of the computer program

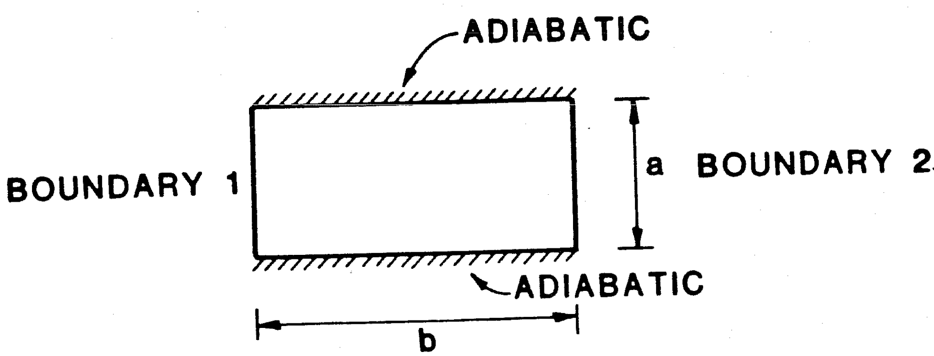


FIG. 2
First test problem configuration

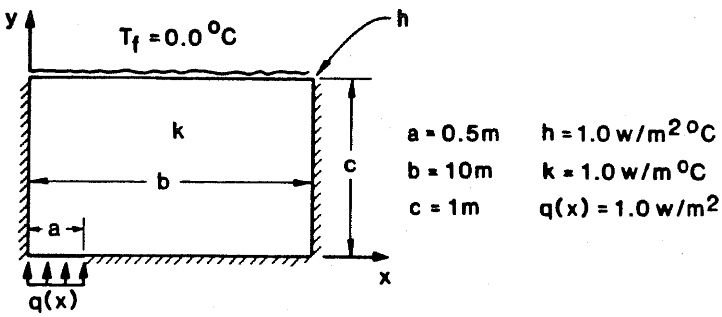


FIG. 3
Second test problem configuration

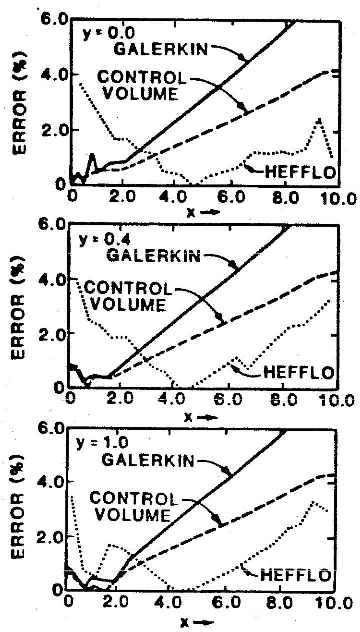


FIG. 4
Comparison of percent error for the Galerkin and finite element control volume formulation versus HEFFLO for the second test problem

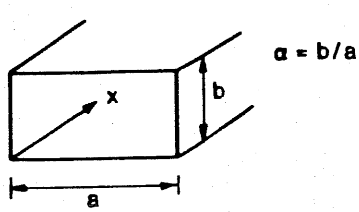


FIG. 5
Third test problem configuration